Unitary Units of The Group Algebra $\mathbb{F}_{2^k}Q_8$

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Abstract: The structure of the unitary unit group of the group algebra $\mathbb{F}_{2^k}Q_8$ is described as a Hamiltonian group.

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1 Introduction

Let KG denote the group ring of the group G over the field K. The homomorphism $\varepsilon : KG \longrightarrow K$ given by $\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$ is called the augmentation mapping of KG. The normalized unit group of KG denoted by $V(KG)$ consists of all the invertible elements of RG of augmentation 1. For further details and background see Polcino Milies and Sehgal [6].

The map $* : KG \longrightarrow KG$ defined by $\left(\sum_{g\in G} a_g g\right)^* = \sum_{g\in G} a_g g^{-1}$ is an anti-automorphism of KG of order 2. An element v of $V(KG)$ satisfying $v^{-1} = v^*$ is called unitary. We denote by $V_*(KG)$ the subgroup of $V(KG)$ formed by the unitary elements of KG.

Let $char(K)$ be the characteristic of the field K. In [2], A.Bovdi and A. Szákacs construct a basis for $V_*(KG)$ where $char(K) > 2$. Also A. Bovdi and L. Erdei [1] determine the structure of $V_*(\mathbb{F}_2G)$ for all groups of order 8 and 16 where $\mathbb{F}_2$ is the Galois field of 2 elements . Additionally in [3], V. Bovdi and A.L. Rosa determine the order of $V_*(\mathbb{F}_{2^k}G)$ for special cases of G. We establish

the structure of $V_*(\mathbb{F}_{2^k}Q_8)$ to be ${C_2}^{4k-1} \times Q_8$ where $Q_8 = \langle x, y \,|\, x^4 = 1, x^2 = y^2, xy = y^{-1}x\rangle$ is the quaternion group of order 8.

1.1 Background

Definition 1.1. *A circulant matrix over a ring R is a square $n \times n$ matrix, which takes the form*

$$circ(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_n & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \ldots & a_1 \end{pmatrix}$$

where $a_i \in R$.

For further details on circulant matrices see Davis [4].

Let $\{g_1, g_2, \ldots, g_n\}$ be a fixed listing of the elements of a group G. Then the following matrix:

$$\begin{pmatrix} {g_1}^{-1}g_1 & {g_1}^{-1}g_2 & {g_1}^{-1}g_3 & \ldots & {g_1}^{-1}g_n \\ {g_2}^{-1}g_1 & {g_2}^{-1}g_2 & {g_2}^{-1}g_3 & \ldots & {g_2}^{-1}g_n \\ {g_3}^{-1}g_1 & {g_3}^{-1}g_2 & {g_3}^{-1}g_3 & \ldots & {g_3}^{-1}g_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {g_n}^{-1}g_1 & {g_n}^{-1}g_2 & {g_n}^{-1}g_3 & \ldots & {g_n}^{-1}g_n \end{pmatrix}$$

is called the matrix of G (relative to this listing) and is denoted by $M(G)$. Let $w = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$ where R is a ring. Then the following matrix:

$$\begin{pmatrix} \alpha_{{g_1}^{-1}g_1} & \alpha_{{g_1}^{-1}g_2} & \alpha_{{g_1}^{-1}g_3} & \ldots & \alpha_{{g_1}^{-1}g_n} \\ \alpha_{{g_2}^{-1}g_1} & \alpha_{{g_2}^{-1}g_2} & \alpha_{{g_2}^{-1}g_3} & \ldots & \alpha_{{g_2}^{-1}g_n} \\ \alpha_{{g_3}^{-1}g_1} & \alpha_{{g_3}^{-1}g_2} & \alpha_{{g_3}^{-1}g_3} & \ldots & \alpha_{{g_3}^{-1}g_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{{g_n}^{-1}g_1} & \alpha_{{g_n}^{-1}g_2} & \alpha_{{g_n}^{-1}g_3} & \ldots & \alpha_{{g_n}^{-1}g_n} \end{pmatrix}$$

is called the RG-matrix of w and is denoted by $M(RG, w)$. The following theorems can be found in [5].

Theorem 1.2. *Given a listing of the elements of a group G of order n there is a ring isomorphism between RG and the $n \times n$ G-matrices over R. This ring isomorphism is given by $\sigma : w \mapsto M(RG, w)$. Suppose R has an identity. Then $w \in RG$ is a unit if and only if $\sigma(w)$ is a unit in $M_n(R)$.*

Example 1.3. *Let* $Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, xy = y^{-1}x\rangle$ *and* $\kappa = \sum_{i=0}^{3} a_i x^i + \sum_{j=0}^{3} b_j x^j y \in \mathbb{F}_{2^k}Q_8$ *where* $a_i, b_j \in \mathbb{F}_{2^k}$*. Then*

$$\sigma(\kappa) = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$$

where $A = circ(a_0, a_1, a_2, a_3)$*,* $B = circ(b_0, b_1, b_2, b_3)$ *and* $C = circ(b_2, b_1, b_0, b_3)$*.*

It is important to note that if $\kappa = \sum_{i=0}^{3} a_i x^i + \sum_{j=0}^{3} b_j x^j y \in \mathbb{F}_{2^k}Q_8$ where $a_i, b_j \in \mathbb{F}_{2^k}$, then $\sigma(\kappa^*) = (\sigma(\kappa))^T$.

The next result can be found in [3]

Proposition 1.4. *Let* K *be a finite field of characteristic* 2*. If* $Q_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1,\ a^{2^{n-1}} = b^2,\ a^b = a^{-1}\rangle$ *is the quaternion group of order* 2^{n+1}*, then*

$$|V_*(KQ_{2^{n+1}})| = 4 \cdot |K|^{2^n}.$$

2 The Structure of The Unitary Subgroup of $\mathbb{F}_{2^k}Q_8$

Proposition 2.1. $Z(V_*(\mathbb{F}_{2^k}Q_8)) \cong {C_2}^{4k}$ *where* $Z(V_*(\mathbb{F}_{2^k}Q_8))$ *is the center of* $V_*(\mathbb{F}_{2^k}Q_8)$*.*

Proof. Let $v = \sum_{i=0}^{3} a_i x^i + \sum_{j=0}^{3} b_j x^j y \in V$ where $V = V(\mathbb{F}_{2^k}Q_8)$ and $a_i, b_j \in \mathbb{F}_{2^k}$. $C_V(x) = \{v \in V \mid xv = vx\}$. Then $xv - vx = (b_3 - b_1)(y) + (b_0 - b_2)xy + (b_1 - b_3)x^2y + (b_2 - b_0)x^3y$. If $\kappa = \sum_{l=0}^{3} c_l x^l + d_1(y + x^2y) + d_2(xy + x^3y)$ where $\sum_{l=0}^{3} c_l = 1$ and $d_j \in \mathbb{F}_{2^k}$, then $\kappa x = x\kappa$. Thus every element of $C_V(x)$ has the form $\sum_{i=0}^{3} a_i x^i + \gamma_1(y + x^2y) + \gamma_2(xy + x^3y)$ where $\sum_{i=0}^{3} a_i = 1$ and $\gamma_j \in \mathbb{F}_{2^k}$.

$Z(V)$ is contained in $C_V(x)$. Therefore $Z(V) = \{\alpha \in C_V(x) \mid \alpha v = v\alpha \text{ for all } v \in V\}$. Let $\alpha = \sum_{i=0}^{3} a_i x^i + b_1(y + x^2 y) + b_2(xy + x^3 y) \in C_V(x)$ and $v = \sum_{l=0}^{3} c_l x^l + \sum_{m=0}^{3} d_m x^m y \in V$ where $a_i, b_j, c_l, d_m \in \mathbb{F}_{2^k}$. Then

$$\begin{aligned}
\sigma(\alpha)\sigma(v) - \sigma(v)\sigma(\alpha) &= \begin{pmatrix} A & B \\ B & A^T \end{pmatrix} \begin{pmatrix} C & D \\ E & C^T \end{pmatrix} - \begin{pmatrix} C & D \\ E & C^T \end{pmatrix} \begin{pmatrix} A & B \\ B & A^T \end{pmatrix} \\
&= \begin{pmatrix} 0 & D(A - A^T) \\ E(A^T - A) & 0 \end{pmatrix}
\end{aligned}$$

where $A = \mathrm{circ}(a_0, a_1, a_2, a_3)$, $B = \mathrm{circ}(b_0, b_1, b_0, b_1)$, $C = \mathrm{circ}(c_0, c_1, c_2, c_3)$, $D = \mathrm{circ}(d_0, d_1, d_2, d_3)$ and $E = \mathrm{circ}(d_2, d_1, d_0, d_3)$, since circulant matrices commute and $B(E - D) = 0 = B(C - C^T)$.
Therefore $\sigma(\alpha)\sigma(v) - \sigma(v)\sigma(\alpha) = 0$ if $D(A - A^T) = 0$ and $E(A^T - A) = 0$. It can be shown that $D(A - A^T) = 0$ and $E(A^T - A) = 0$ iff $a_1 = a_3$. Thus every element of the $Z(V)$ has the form $1 + r + sx + rx^2 + sx^3 + ty + uxy + tx^2 y + ux^3 y$ where $r, s, t, u \in \mathbb{F}_{2^k}$. It can easily be shown that $Z(V)$ has exponent 2.

Now $\alpha^* = \alpha^{-1} \iff \sigma(\alpha^*) = \sigma(\alpha^{-1}) \iff (\sigma(\alpha))^T = \sigma(\alpha)^{-1} \iff \sigma(\alpha)(\sigma(\alpha))^T = I$. Let $\alpha = 1 + r + sx + rx^2 + sx^3 + ty + uxy + tx^2 y + ux^3 y \in Z(V)$ where $r, s, t, u \in \mathbb{F}_{2^k}$. Then

$$\sigma(\alpha)(\sigma(\alpha))^T = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} A & B \\ B & A \end{pmatrix}^T = \begin{pmatrix} A^2 + B^2 & 0 \\ 0 & A^2 + B^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

where $A = \mathrm{circ}(1 + r, s, r, s)$, $B = \mathrm{circ}(t, u, t, u)$. Therefore $Z(V) \subset V_*(\mathbb{F}_{2^k} Q_8)$.

Thus $Z(V_*(\mathbb{F}_{2^k} Q_8)) = Z(V)$ and $Z(V_*(\mathbb{F}_{2^k} Q_8)) \cong {C_2}^{4k}$. $\square$

We can now construct the following subgroup lattice of $V_*(\mathbb{F}_{2^k} Q_8)$:

$$\begin{tikzcd}
& V_*(\mathbb{F}_{2^k} Q_8) Q_8 \arrow[dl, equal] \arrow[dr, equal] & \\
Z(V_*(\mathbb{F}_{2^k} Q_8)) \arrow[dr, equal] & & Q_8 \arrow[dl, equal] \\
& Z(V_*(\mathbb{F}_{2^k} Q_8)) \cap Q_8 = \{1, x^2\} \arrow[d, equal] & \\
& 1 &
\end{tikzcd}$$

Proposition 2.2. $Z(V_*(\mathbb{F}_{2^k} Q_8)).Q_8 = V_*(\mathbb{F}_{2^k} Q_8)$.

Proof. By the second isomorphism theorem $Z(V_*(\mathbb{F}_{2^k}Q_8)).Q_8/Z(V_*(\mathbb{F}_{2^k}Q_8)) \cong Q_8/Z(V_*(\mathbb{F}_{2^k}Q_8)) \cap Q_8$. $|Q_8/Z(V_*(\mathbb{F}_{2^k}Q_8)) \cap Q_8| = \dfrac{8}{2} = 4$. Therefore $|Z(V_*(\mathbb{F}_{2^k}Q_8)).Q_8| = 4.2^{4k} = 2^{4k+2}$. Therefore $Z(V_*(\mathbb{F}_{2^k}Q_8)).Q_8 = V_*(\mathbb{F}_{2^k}Q_8)$. $\square$

Theorem 2.3. $V_*(\mathbb{F}_{2^k}Q_8) \cong {C_2}^{4k-1} \times Q_8$.

Proof. $Z(V_*(\mathbb{F}_{2^k}Q_8)) \cong {C_2}^{4k}$ is a vector space over $\mathbb{F}_2$ of dimension $4k$. Let $\{x_1, x_2, \ldots, x_{4k} = x^2\}$ be a basis for this vector space. Therefore $Z(V_*(\mathbb{F}_{2^k}Q_8)) = \langle x_1, x_2, \ldots, x_{4k}\rangle$. Let $G = \langle x_1, x_2, \ldots, x_{4k-1}\rangle$, then $G \cong {C_2}^{4k-1}$ and $Z(V_*(\mathbb{F}_{2^k}Q_8)) \cong G \times \langle x_{4k}\rangle \cong G \times \langle x^2\rangle$. Now $G \cap Q_8 = \{1\}$ and $V_*(\mathbb{F}_{2^k}Q_8) = G.Q_8$, therefore $V_*(\mathbb{F}_{2^k}Q_8) \cong G \rtimes Q_8 \cong G \times Q_8$ since $G < Z(V_*(\mathbb{F}_{2^k}Q_8))$. Thus $V_*(\mathbb{F}_{2^k}Q_8) \cong {C_2}^{4k-1} \times Q_8$. $\square$

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